How to glue perverse sheaves

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The aim of this note [0] is to give a short, self-contained account of the vanishing cycle constructions of perverse sheaves; e.g., for the needs of [1]. It differs somewhat from the alternative approaches offered by MacPherson–Vilonen [6] and Verdier [8, 9], which justifies, possibly, its publication.

The text follows closely the notes written down by S. I. Gel'fand in 1982. I am much obliged to him. I am also much obliged to J. Bernstein: the construction of the functor Ψ in n°2 below was found in a joint work with him in spring 1981.

We will consider the algebraic situation only; for notations see [1, §1].

1. The monodromy Jordan block In this preliminary we will fix the notation concerning the standard local systems on $\mathbb{A}^1 \setminus \{0\}$ with unipotent monodromy of a single Jordan block.

1.1. Let us start with the classical topology situation, so in this n°, $k = \mathbb{C}$. For an integer i, as usual,

$$\mathbb{Z}(i) := (2\pi\sqrt{-1})^i\mathbb{Z} = \mathbb{Z}(1)^{\otimes i}, \quad \mathbb{Z}(1) = \pi_1\big((\mathbb{A}^1 \setminus \{0\})(\mathbb{C}), 1\big).$$

Consider the group algebra $\mathbb{Z}[\mathbb{Z}(1)]$; for $l \in \mathbb{Z}(1)$, let $\widetilde{l}$ denote l as an (invertible) element in $\mathbb{Z}[\mathbb{Z}(1)]$. Let

$$I = \mathbb{Z}[\mathbb{Z}(1)](\widetilde{t} - 1), \text{ where } t \text{ is a generator of } \mathbb{Z}(1)$$

be the augmentation ideal. Let A^0 be the I-adic completion of $\mathbb{Z}[\mathbb{Z}(1)]$:

$$A^0 = \mathbb{Z}[\![\widetilde{t} - 1]\!], \quad A^i = (\widetilde{t} - 1)^i A^0 = (A^1)^i \quad (i \geq 0).$$

The graded ring $\mathrm{Gr}\, A^0 = \bigoplus_{i\geq 0} A^i/A^{i+1}$ is canonically isomorphic to the polynomial ring $\mathbb{Z}\oplus\mathbb{Z}(1)\oplus\mathbb{Z}(2)\oplus\cdots$. Put

$$(A^i)^* = \{x \in A^i \mid A^i = xA^0\} = \{x \in A^i \mid x \bmod A^{i+1} \text{ generates } A^i/A^{i+1} = \mathbb{Z}(i)\};$$

one has $(A^i)^* \cdot (A^j)^* = (A^{i+j})^*$, so $\bigcup_{i\geq 0}(A^i)^*$ is a multiplicative system. Let $A \supset A^0$ be the corresponding localization of A^0; one has

$$A = A^0_{(\widetilde{t}-1)} = \mathbb{Z}((\widetilde{t} - 1)).$$

The ring A has a natural $\mathbb{Z}$-filtration

$$A^i = [(A^1)^*]^i A^0 = (\widetilde{t} - 1)^i A^0;$$

for $i \geq 0$ these A^i coincide with the ones above; one has $\mathrm{Gr}\, A = \bigoplus_{i\in\mathbb{Z}} A^i/A^{i+1} = \bigoplus \mathbb{Z}(i)$, a Laurent polynomial ring.

Define a $\mathbb{Z}$-bilinear pairing $\langle\cdot,\cdot\rangle\colon A \times A \to \mathbb{Z}(-1)$ by the formula

$$(\langle f, g\rangle, t) = \mathrm{Res}_{\widetilde{t}=1}(fg^- d\log\widetilde{t})$$

where $g \mapsto g^-$ is the canonical involution of the ring A, $\widetilde{l}^- = (\widetilde{l})^{-1} = \widetilde{(-l)}$. This pairing is skew-symmetric and $\mathbb{Z}(1)$-invariant:

$$\langle f, g\rangle = -\langle g, f\rangle, \quad \langle lf, lg\rangle = \langle f, g\rangle;$$

it is compatible with the filtration and non-degenerate:

$$A^1 = (A^{-1})^\perp, \quad A^{a,b} := A^a/A^b \xrightarrow{\sim} \operatorname{Hom}\big(A^{-b}/A^{-a}, \mathbb{Z}(-1)\big)$$

and the pairing induced on $\operatorname{Gr} A$ is:

$$\langle S_i, S_{-i-1}\rangle = (-1)^i S_i \cdot S_{-i-1}, \text{ where } S_j \in \mathbb{Z}(j).$$

Let $\mathcal{J}$ be the local system of invertible A-modules on $(\mathbb{A}^1 \setminus \{0\})(\mathbb{C})$ such that the fiber of $\mathcal{J}$ over $1 \in \mathbb{A}^1(\mathbb{C})$ is A, and the monodromy action of $l \in \pi_1\big((\mathbb{A}^1 \setminus \{0\})(\mathbb{C}), 1\big)$ is the multiplication by $\widetilde{l}$. We have a canonical filtration $\mathcal{J}^i$ on $\mathcal{J}$, such that

$$(\mathcal{J}^i)_1 = A^i, (\mathcal{J}^i)_1/(\mathcal{J}^{i+1})_1 = \mathbb{Z}(i),$$

together with a non-degenerate skew-symmetric pairing $\langle\cdot,\cdot\rangle\colon \mathcal{J}\times\mathcal{J} \to \underline{\mathbb{Z}}(-1)$, compatible with the filtration, that coincides on $\mathcal{J}_1$ with the above $\langle\cdot,\cdot\rangle$. Put $\mathcal{J}^{a,b} = \mathcal{J}^a/\mathcal{J}^b$. We may consider $\mathcal{J} = \underset{\longleftrightarrow}{\lim}\, \mathcal{J}^{a,b}$ as filtered A-objects in the category

$$\underset{\longleftrightarrow}{\lim}\{\text{local systems on } (\mathbb{A}^1 \setminus \{0\})(\mathbb{C})\}$$

(for the definition of $\underset{\longleftrightarrow}{\lim}$ see the Appendix, A.3).

1.2. These definitions have an obvious étale version: just replace $\mathbb{Z}(i)$ above by $(\mathbb{Z}/\ell^n)(i)$ and repeat 1.1 word-for-word. We get the ring $A_{\text{ét}} = \underset{\longleftrightarrow}{\lim}\, A^{a,b}$ in $\underset{\longleftrightarrow}{\lim}\{\text{sheaves on } (\operatorname{Spec} k)_{\text{ét}}\}$ and the $A_{\text{ét}}$-object $\mathcal{J}_{\text{ét}} = \underset{\longleftrightarrow}{\lim}\, \mathcal{J}^{a,b}_{\text{ét}}$ in $\underset{\longleftrightarrow}{\lim}\{\text{sheaves on } (\mathbb{A}^1 \setminus \{0\})_{\text{ét}}\}$. In the same way we get $\mathbb{Q}_\ell$- and mixed variants.

1.3. The holonomic counterpart of the above is as follows. Put

$$A_{\text{hol}} = k((s)), \quad A^i_{\text{hol}} = s^i k[[s]];$$

define the pairing $\langle\cdot,\cdot\rangle\colon A_{\text{hol}} \times A_{\text{hol}} \to k$ by the formula

$$\langle f(s), g(s)\rangle = \operatorname{Res}_{s=0} f(s)g(-s)ds.$$

This pairing has the same properties as the one above (invariance: $\langle sf, g\rangle + \langle f, sg\rangle = 0$). For integers $a \le b$ let $\mathcal{J}^{a,b}_{\text{hol}}$ be a D-module on $\mathbb{A}^1 \setminus \{0\}_k$ such that $\mathcal{J}^{a,b}_{\text{hol}} \otimes \mathcal{O}$ as an $\mathcal{O}$-module and $\nabla\alpha = s\alpha\frac{dx}{x}$ for $A^{a,b}_{\text{hol}}$ (here x is the parameter on $\mathbb{A}^1 \setminus \{0\} = \mathbf{G_m}$). Put $\mathcal{J}_{\text{hol}} = \underset{\longleftrightarrow}{\lim}\, \mathcal{J}^{a,b}_{\text{hol}}$. This is a filtered A_{hol}-object in $\underset{\longleftrightarrow}{\lim}\, \mathbf{M}_{\text{hol}}(\mathbf{G_m})$. Clearly, $\operatorname{Gr}\mathcal{J} = \bigoplus \mathcal{O}_{\mathbf{G_m}} \cdot s^i$ (recall that the Tate twist is the identity functor in the holonomic situation).

1.4. For any $\pi \in (A^1)^*$ we have isomorphisms

$$\sigma_\pi\colon \mathcal{J}^{a,b} \to \mathcal{J}^{a+n,b+n}(-n), \quad \sigma_\pi(x) = \pi^n x \otimes \overline{\pi}^{-n},$$

where $\overline{\pi} = \pi \bmod A^2$. In the holonomic or in the $\mathbb{Q}_\ell$-situation, we have a canonical choice σ of σ_π: in the holonomic case, put $\sigma = \sigma_s =$ multiplication by s^n; in the $\mathbb{Q}_\ell$-case put $\sigma = \sigma_{\log t}$, where t is a generator of $\mathbb{Q}_\ell(1)$.

In what follows I will consider the $\mathcal{J}^{a,b}$ as ordinary sheaves on $\mathbb{A}^1 \setminus \{0\}$, so $\mathcal{J}^{a,b}$ lives in $\mathbf{M}(\mathbb{A}^1 \setminus \{0\})[-1] \subset \mathbf{D}(\mathbb{A}^1 \setminus \{0\})$.

2. The unipotent nearby cycles functor $\Psi^{\mathbf{un}}$ Let X be a scheme, and $f \in \mathcal{O}(X)$ a fixed function. Put

$$Y := f^{-1}(0) \overset{i}{\hookrightarrow} X \overset{j}{\hookleftarrow} U := X \setminus Y, \quad \mathcal{J}^{a,b}_f := (f|_U)^*(\mathcal{J}^{a,b})$$

(so, in the holonomic case, $\mathcal{J}_f$ is the D-module generated by f^s). For any $\mathcal{M} \in \mathbf{M}(U)$ consider

$$\mathcal{M} \otimes \mathcal{J}_f = \underset{\longleftrightarrow}{\lim}\big(\mathcal{M} \otimes \mathcal{J}^{a,b}_f\big)$$

in $\underset{\longleftrightarrow}{\lim}\, \mathbf{M}(U)$. The ring A acts on $\mathcal{M} \otimes \mathcal{J}_f$ via $\mathcal{J}_f$, and the pairing $\langle\cdot,\cdot\rangle$ defines a canonical isomorphism $\mathbb{D}(\mathcal{M} \otimes \mathcal{J}_f) = \mathbb{D}(\mathcal{M}) \otimes \mathcal{J}_f(1)$ compatible with the A-action.

2.1. Key Lemma The canonical arrow $\alpha\colon j_!(\mathcal{M}\otimes\mathcal{J}_f)\to j_*(\mathcal{M}\otimes\mathcal{J}_f)$ in $\underset{\longleftrightarrow}{\lim}\,\mathbf{M}(X)$ is an isomorphism.

Proof. The lemma would follow if for $\pi\in(A^1)^*$ we could find a certain $N\geq 0$ and a compatible system of morphisms

$$\beta^{a,b}\colon j_*(\mathcal{M}\otimes\mathcal{J}_f^{a,b})\to j_!(\mathcal{M}\otimes\mathcal{J}_f^{a,b})$$

such that $\alpha^{a,b}\circ\beta^{a,b}=\pi^N=\beta^{a,b}\circ\alpha^{a,b}$ (where the $\alpha^{a,b}$ are the (a,b)-components of α); then $\alpha^{-1}=\pi^{-N}\underset{\longleftrightarrow}{\lim}\,\beta^{a,b}$.

To do this, it suffices to show that all $\ker\alpha^{a,b},\operatorname{coker}\alpha^{a,b}$ are annihilated by some π^m independent of (a,b): just take $N=2m$ and $\beta^{a,b}=\beta_!\circ\beta_*$ in the commutative diagram

$$\begin{tikzcd}
j_!(\mathcal{M}\otimes\mathcal{J}^{a,b}) \arrow[dr] \arrow[dd, "\pi^m"'] & & j_*(\mathcal{M}\otimes\mathcal{J}^{a,b}) \arrow[dl, dashed, "\beta_*"'] \arrow[dd, "\pi^m"] \\
& \operatorname{coim}\alpha^{a,b}=\operatorname{im}\alpha^{a,b} \arrow[dl, dashed, "\beta_!"] \arrow[dr, hook] & \\
j_!(\mathcal{M}\otimes\mathcal{J}^{a,b}) & & j_*(\mathcal{M}\otimes\mathcal{J}^{a,b})
\end{tikzcd}$$

By duality we may consider coker's only. In the holonomic case the desired fact follows from the lemma on b-functions (see [2, §3.8]), since $\mathcal{M}\otimes\mathcal{J}_f=\mathcal{M}\cdot f^s((s))$: take $m=\sum l(u)$, where u runs over a finite set of generators of $\mathcal{M}$ and $l(u)$ is the number of integral roots of the b-function of u. In the constructible case one should use the finiteness theorem for the usual nearby cycles functor $R\Psi$ (see [4, §3]). Note that for any $\mathcal{F}$ in $\mathbf{D}(X)$ we have a distinguished triangle

$$i^*j_*\mathcal{F}\to R\Psi^{\mathrm{un}}(\mathcal{F})\xrightarrow{1-t}R\Psi^{\mathrm{un}}(\mathcal{F})\to$$

in $\mathbf{D}(Y)$, where $R\Psi^{\mathrm{un}}$ is the part of $R\Psi$ on which monodromy acts in a unipotent way, and t is a generator of the monodromy group $\mathbb{Z}_\ell(1)$. Therefore

$$\begin{aligned}\operatorname{Cone}\bigl(j_!(\mathcal{M}\otimes\mathcal{J}_f^{a,b})\to j_*(\mathcal{M}\otimes\mathcal{J}_f^{a,b})\bigr)&=i^*j_*(\mathcal{M}\otimes\mathcal{J}_f^{a,b})\\&=\operatorname{Cone}\bigl(R\Psi^{\mathrm{un}}(\mathcal{M})\otimes\mathcal{J}_1^{a,b}\xrightarrow{1-t}R\Psi^{\mathrm{un}}(\mathcal{M})\otimes\mathcal{J}_1^{a,b}\bigr)[-1],\end{aligned}$$

where t acts on $R\Psi^{\mathrm{un}}(\mathcal{M})\otimes\mathcal{J}_1^{a,b}=R\Psi^{\mathrm{un}}(\mathcal{M})\otimes A^{a,b}$ via $t\otimes\widetilde{t}$. Since $A^{a,b}$ is a $\mathbb{Z}_\ell(1)$-module with one generator, the power of $1-t$ that annihilates $R\Psi^{\mathrm{un}}(\mathcal{M})$ annihilates the cone also, and we are done by the finiteness theorem for $R\Psi$ (see [4, §3]). $\square$

2.2.[*] Put $\Pi_f(\mathcal{M}):=j_!(\mathcal{M}\otimes\mathcal{J}_f)=j_*(\mathcal{M}\otimes\mathcal{J}_f)$; clearly $\Pi_f\colon\mathbf{M}(U)\to\underset{\longleftrightarrow}{\lim}\,\mathbf{M}(X)$ is an exact functor (since so are $j_!$ and j_*), and $\langle\cdot,\cdot\rangle$ defines a canonical isomorphism $\mathbb{D}\,\Pi_f(\mathcal{M})=\Pi_f(\mathbb{D}\,\mathcal{M})(1)$. On Π_f, there are two admissible filtrations:

$$\begin{aligned}\Pi_!^\bullet(\mathcal{M})&=j_!(\mathcal{M}\otimes\mathcal{J}_f^\bullet)\\\Pi_*^\bullet(\mathcal{M})&=j_*(\mathcal{M}\otimes\mathcal{J}_f^\bullet);\end{aligned}$$

one has $\Pi_!^\bullet\subset\Pi_*^\bullet$, $\operatorname{Gr}^\bullet_{\Pi_!}(\mathcal{M})=j_!\mathcal{M}(\cdot)$, $\operatorname{Gr}^\bullet_{\Pi_*}(\mathcal{M})=j_*\mathcal{M}(\cdot)$. By 2.1, for $a\leq b$ any

$$\Pi_{!*}^{a,b}(\mathcal{M}):=\Pi_*^a(\mathcal{M})/\Pi_!^b(\mathcal{M})$$

belongs to $\mathbf{M}(X)\subset\underset{\longleftrightarrow}{\lim}\,\mathbf{M}(X)$. Clearly the $\Pi_{!*}^{a,b}\colon\mathbf{M}(U)\to\mathbf{M}(X)$ are exact functors, $\Pi=\underset{\longleftrightarrow}{\lim}\,\Pi_{!*}^{a,b}$; one has $\mathbb{D}\,\Pi_{!*}^{a,b}(\mathcal{M})=\Pi_{!*}^{-b,-a}(\mathbb{D}\,\mathcal{M})(1)$; and (see 1.4) we have isomorphisms $\sigma_\pi\colon\Pi_{!*}^{a,b}\xrightarrow{\sim}\Pi_{!*}^{a+n,b+n}(-n)$.

[*]The following constructions are quite parallel to the Lax-Phillips scheme in scattering theory: the multiplication by s is time translation, $\Pi_!^0,\Pi/\Pi_*^0$ are in- and out-spaces, etc.

2.3. We will need the following particular $\Pi^{a,b}_{!*}$-functors. The first is the functor Ψ^{un}_f or simply Ψ_f for short, of *unipotent nearby cycles*, and its relatives $\Psi^{(i)}_f$:

$$\Psi^{\mathrm{un}}_f := \Pi^{0,0}_{!*}, \quad \Psi^{(i)}_f := \Pi^{i,i}_{!*} \xleftarrow[\sim]{\sigma_\pi} \Psi_f(i) .$$

These take values in $\mathbf{M}(Y) \subset \mathbf{M}(X)$, and we have $\Psi^{(i)}_f \mathbb{D} = \mathbb{D}\,\Psi^{(-i)}_f(1)$. The second is Ξ_f, the *maximal extension functor*, and the corresponding Ξ^i_f:

$$\Xi_f := \Pi^{0,1}_{!*}, \quad \Xi^i_f := \Pi^{i,i+1}_{!*} \xleftarrow[\sim]{\sigma_\pi} \Xi_f(i) .$$

We have canonical exact sequences

$$0 \to j_!(\mathcal{M})(a) \xrightarrow{\alpha_-} \Xi^a_f(\mathcal{M}) \xrightarrow{\beta_-} \Psi^{(a)}_f(\mathcal{M}) \to 0$$
$$0 \to \Psi^{(a+1)}_f(\mathcal{M}) \xrightarrow{\beta_+} \Xi^a_f(\mathcal{M}) \xrightarrow{\alpha_+} j_*(\mathcal{M})(a) \to 0,$$

which are interchanged by duality. Here $\alpha_+ \circ \alpha_- = \alpha$ is the canonical morphism $j_! \to j_*$, and $\beta_- \circ \beta_+ = \beta\colon \Psi^{(1)}_f \to \Psi^{(0)}_f$ is the canonical arrow $\Pi^{1,1}_{!*} \to \Pi^{0,0}_{!*}$; under the isomorphism $\sigma_\pi\colon \Psi^{(0)}_f \to \Psi^{(1)}_f(-1)$ the arrow becomes multiplication by $\overline{\pi}^{-1}$.

3. Vanishing cycles and the gluing functor Let $\mathcal{M}_X$ be a perverse sheaf on X, $\mathcal{M}_U$ its restriction to U. Consider the following complex,

$$j_!(\mathcal{M}_U) \xrightarrow{(\alpha_-,\gamma_-)} \Xi_f(\mathcal{M}_U) \oplus \mathcal{M}_X \xrightarrow{(\alpha_+,-\gamma_+)} j_*(\mathcal{M}_U) \tag{$*$}$$

where $\alpha_\pm, \gamma_\pm$ are the (only) arrows that coincide with $\mathrm{id}_{\mathcal{M}_U}$ on U. Put

$$\Phi_f(\mathcal{M}_X) := \ker(\alpha_+, -\gamma_+)/\operatorname{im}(\alpha_-, \gamma_-).$$

Clearly $\Phi_f(\mathcal{M}_X)$ is supported on Y and $\Phi_f\colon \mathbf{M}(X) \to \mathbf{M}(Y)$ is an exact functor (since α_- is injective and α_+ is surjective). We have canonical arrows

$$\Psi^{(1)}_f(\mathcal{M}_U) \xrightarrow{u} \Phi_f(\mathcal{M}_X) \xrightarrow{v} \Psi_f(\mathcal{M}_U)$$

given by the formulas

$$u(\psi) = (\beta_+(\psi), 0), \quad v(\xi, m) = \beta_-(\xi);$$

clearly $v \circ u = \beta_- \circ \beta_+ = \beta$.

Define a *vanishing cycles data* for f, or *f-data* for short, to be a quadruple $(\mathcal{M}_U, \mathcal{M}_Y, u, v)$, with $\mathcal{M}_U \in \mathbf{M}(U)$, $\mathcal{M}_Y \in \mathbf{M}(Y)$, and $\Psi^{(1)}_f(\mathcal{M}_U) \xrightarrow{u} \mathcal{M}_Y \xrightarrow{v} \Psi_f(\mathcal{M}_U)$ such that $v \circ u = \beta$. The f-data form an abelian category $\mathbf{M}_f(U, Y)$ in the obvious way. Put

$$F_f(\mathcal{M}_X) := \big(\mathcal{M}_U, \Phi_f(\mathcal{M}_X), u, v\big);$$

clearly this defines an exact functor $F_f\colon \mathbf{M}(X) \to \mathbf{M}_f(U, Y)$. Conversely, let $(\mathcal{M}_U, \mathcal{M}_Y, u, v)$ be f-data. Consider the complex

$$\Psi^{(1)}_f(\mathcal{M}_U) \xrightarrow{(\beta_+,u)} \Xi_f(\mathcal{M}_U) \oplus \mathcal{M}_Y \xrightarrow{(\beta_-,-v)} \Psi_f(\mathcal{M}_U). \tag{$**$}$$

Put

$$G_f(\mathcal{M}_U, \mathcal{M}_Y, u, v) = \ker(\beta_-, -v)/\operatorname{im}(\beta_+, u) :$$

this defines an exact functor $G_f\colon \mathbf{M}_f(U, Y) \to \mathbf{M}(X)$ (G_f is exact since β_+ is mono and β_- is epi). Call G_f the *gluing functor*.

3.1. Proposition The functors $\mathbf{M}(X) \underset{G_f}{\overset{F_f}{\rightleftarrows}} \mathbf{M}_f(U,Y)$ are mutually inverse equivalences of categories.

Proof. For $\mathcal{M}_X$ in $\mathbf{M}(X)$ consider $(*)$ as a diad (for diads see the Appendix, A.2); this way we may identify $\mathbf{M}(X)$ with the category of diads of type $(*)$ having the property that both $\gamma_+|_U$ and $\gamma_-|_U$ are isomorphisms. In the same way, for $(\mathcal{M}_U, \mathcal{M}_Y, u, v) \in \mathbf{M}_f(U,Y)$ take the diad $(**)$; in this way we identify $\mathbf{M}_f(U,Y)$ with the category of diads of type $(**)$ having the property that $\mathcal{M}_Y$ is supported on Y. After this identification is done, we see that F_f and G_f are just the reflection functors; since $r \circ r = \mathrm{id}$, we are done. $\square$

Here is the simplest case of the above proposition:

3.2. Corollary Let X be a small disk around 0 in the complex line. Then the category $\mathbf{M}(X)$ of perverse sheaves on X (with singularities at 0 only) is equivalent to the category $\mathbf{C}$ of diagrams $V_0 \underset{u}{\overset{v}{\rightleftarrows}} V_1$ of vector spaces such that both operators $\mathrm{id}_{V_0} - (u \circ v)$ and $\mathrm{id}_{V_1} - (v \circ u)$ are invertible.

Proof. For a vector space V and $\phi \in \operatorname{End} V$, let $(V, \phi)^0 \subset V$ be the maximal subspace on which ϕ acts in a nilpotent way. Consider the category $\mathbf{C}'$ of diagrams (V_0', V_1', ϕ, u, v), where V_0', V_1' are vector spaces, $\phi \in \operatorname{Aut} V_1'$, and $(V_1', \mathrm{id}_{V_1'} - \phi)^0 \underset{v}{\overset{u}{\rightleftarrows}} V_0'$ are such that $v \circ u = \mathrm{id} - \phi$. The category $\mathbf{C}$ is equivalent to $\mathbf{C}'$ via the functor

$$(V_0 \underset{u}{\overset{v}{\rightleftarrows}} V_1) \mapsto \big((V_0, u \circ v)^0, V_1, \mathrm{id}_{V_1} - (v \circ u), u, v\big).$$

The category $\mathbf{M}_f(X \setminus \{0\}, \{0\})$, where $f \colon X \hookrightarrow \mathbb{A}^1$, is equivalent to $\mathbf{C}'$, since $M(\{0\}) = \{\text{vector spaces}\}$, $\mathbf{M}(U) = \{\text{vector spaces with an automorphism (monodromy)}\}$, and, under this identification, $\Psi_f\big((V, \phi)\big) = (V, \mathrm{id}_V - \phi)^0$. Now apply 3.1. $\square$

Remark The end of the proof of 2.1 in fact shows that $\Psi = \Psi^{\mathrm{un}}$ as defined in 2.3 coincides with the standard $R\Psi^{\mathrm{un}}[-1]$; the same is true for Φ. One may recover all $R\Psi(\mathcal{M})$ by applying Ψ^{un} to $\mathcal{M} \otimes f^*(?)$, where ? runs through the irreducible local systems on a punctured disk. For example, if $\mathcal{M}$ is RS holonomic, then the component of $R\Psi(DR(\mathcal{M}))$ that corresponds to the eigenvalue $\alpha \in \mathbb{C}^*$ of the monodromy is just $(DR)\Psi^{\mathrm{un}}(\mathcal{M} \cdot f^a)$, where $\exp(2\pi i a) = \alpha$ (since Ψ^{un}, obviously, commutes with DR). This fact was also found by Malgrange [7] and Kashiwara [5]. See also [3] where the total nearby cycles functor for arbitrary holonomic modules (not necessarily RS) was introduced.

A. Appendix Here some linear algebra constructions, needed in the main body of the paper, are presented. Below, $\mathbf{A}$ will be an exact category in the sense of Quillen; as usual, $\hookrightarrow, \twoheadrightarrow$ denote an admissible monomorphism, resp. epimorphism. If $\mathbf{C}$ is any category, then $\mathbf{C}^\circ$ is its dual.

A.1. Monads A *monad* is a complex of the form

$$\mathcal{P} = \big(P_- \overset{\alpha_-}{\rightarrowtail} P \overset{\alpha_+}{\twoheadrightarrow} P_+.\big)$$

Denote by $H(\mathcal{P}) := \ker\alpha_+/\operatorname{im}\alpha_- \in \operatorname{Ob}(\mathbf{A})$ the cohomology of $\mathcal{P}$. The category of monads $\widetilde{\mathbf{A}}$ is an exact category: the exact sequences in $\widetilde{\mathbf{A}}$ are the ones which are componentwise exact; it is easy to see that $H\colon \widetilde{\mathbf{A}} \to \mathbf{A}$ is an exact functor. An exact functor between exact categories induces one between their categories of monads; these functors commute with H. Also one has $(\widetilde{\mathbf{A}})^\circ = \widetilde{\mathbf{A}^\circ}$.

Often it is convenient to represent monads by somewhat different types of diagrams. Namely, let $\widetilde{\mathbf{A}}_1$ be the category of objects together with a 3-step admissible filtration

$$\mathcal{P}_1 = \big(\ P_{-1} \overset{\gamma_{-1}}{\rightarrowtail} P_0 \overset{\gamma_0}{\rightarrowtail} P_1\ \big);$$

and $\widetilde{\mathbf{A}}_2$ be the category of short exact sequences

$$\mathcal{P}_2 = \big(\ L_- \overset{(\delta_-,\varepsilon_-)}{\rightarrowtail} A \oplus B \overset{(\delta_+,\varepsilon_+)}{\twoheadrightarrow} L_+\ \big)$$

such that δ_- is an admissible monomorphism, ε_- an admissible epimorphism. These are exact categories in the same way as $\widetilde{\mathbf{A}}$.

Lemma The categories $\widetilde{\mathbf{A}}$, $\widetilde{\mathbf{A}}_1$, $\widetilde{\mathbf{A}}_2$ are canonically equivalent.

Proof. Here are the corresponding functors: the functor $\widetilde{\mathbf{A}} \to \widetilde{\mathbf{A}}_1$ maps $\mathcal{P}$ above to $P_- \hookrightarrow \ker\alpha_+ \hookrightarrow P$; the one $\widetilde{\mathbf{A}}_1 \to \widetilde{\mathbf{A}}_2$ maps $\mathcal{P}_1$ above to $P_0 \overset{(\gamma_0,\varepsilon_-)}{\rightarrowtail} P_1 \oplus P_0/P_{-1} \overset{(\delta_+,-\gamma_0)}{\twoheadrightarrow} P_1/P_{-1}$ where ε_-, δ_+ are the natural projections; finally, $\widetilde{\mathbf{A}}_2 \to \widetilde{\mathbf{A}}$ maps $\mathcal{P}_2$ above to $\ker\varepsilon_- \hookrightarrow A \twoheadrightarrow A/\delta_-(L_-)$. I leave the proof of the lemma to the reader; note that B in the $\widetilde{\mathbf{A}}_2$-avatar is $H(\mathcal{P})$. $\square$

A.2. Diads and the reflection functor A *diad* in $\mathbf{A}$ is a commutative diagram of the form

$$\mathcal{Q} = \left(\begin{tikzcd} & A \arrow[dr, "\alpha_+"] & \\ C_- \arrow[ur, tail, "\alpha_-"] \arrow[dr, "\beta_-"'] & & C_+ \\ & B \arrow[ur, "\beta_+"'] & \end{tikzcd}\right).$$

Clearly, diads with componentwise exact sequences form an exact category $\mathbf{A}^\#$; one has $(\mathbf{A}^\#)^\circ = (\mathbf{A}^\circ)^\#$ and exact functors between $\mathbf{A}$'s induce ones betwen $\mathbf{A}^\#$'s.

As in the case of monads, we may represent diads by other diagrams. Let $\mathbf{A}_1^\#$ be the category of monads of the form

$$\mathcal{Q}_1 = \big(\ C_- \rightarrowtail A \oplus B \twoheadrightarrow C_+\ \big)$$

such that the corresponding arrow $C_- \to A$ is an admissible monomorphism, and the one $A \to C_+$ is an admissible epimorphism. Let $\mathbf{A}_2^\#$ be the category of short exact sequences

$$\mathcal{Q}_2 = \big(\ D_- \overset{(\gamma_-,\delta_-^1,\delta_-^2)}{\rightarrowtail} A \oplus B^1 \oplus B^2 \overset{(\gamma_+,\delta_+^1,\delta_+^2)}{\twoheadrightarrow} D_+\ \big)$$

such that both $D_- \xrightarrow{(\gamma_-,\delta_-^i)} A \oplus B^i$ are admissible monomorphisms, and both $A \oplus B^i \xrightarrow{(\gamma_+,\delta_+^i)} D_+$ are admissible epimorphisms. These are exact categories in the same way as $\mathbf{A}^\#$.

Lemma The categories $\mathbf{A}^{\#}$, $\mathbf{A}_1^{\#}$, and $\mathbf{A}_2^{\#}$ are canonically equivalent.

Proof. The corresponding functors are: the one $\mathbf{A}^{\#} \to \mathbf{A}_1^{\#}$ maps $\mathcal{Q}$ above to

$$C_- \xrightarrow{(\alpha_-,-\beta_-)} A \oplus B \xrightarrow{(\alpha_+,\beta_+)} C_+$$

and the one $\mathbf{A}_1^{\#} \to \mathbf{A}_2^{\#}$ maps $\mathcal{Q}_1$ above to its $\widetilde{\mathbf{A}}_2$-avatar (so $B^1 = B, B^2 = H(\mathcal{Q}_1)$). The first functor is obviously an equivalence of categories; as for the second one, this follows from Lemma A.1. $\square$

Note that in the diagram of type $\mathbf{A}_2^{\#}$ one may interchange the objects B^i. This defines the important automorphism r of $\mathbf{A}_2^{\#}$, and thus of $\mathbf{A}^{\#}$, with $r \circ r = \mathrm{id}_{\mathbf{A}^{\#}}$; call it the *reflection functor*. Clearly it transforms a diad $\mathcal{Q}$ above into

$$r(\mathcal{Q}) = \left(\begin{array}{ccccc} & & A & & \\ & \nearrow & & \searrow & \\ \ker \alpha_+ & & & & \operatorname{coker} \alpha_- \\ & \searrow & & \nearrow & \\ & & H(\mathcal{Q}) & & \end{array} \right),$$

where $H(\mathcal{Q}) := H(C_- \to A \oplus B \to C_+)$ is the cohomology of the corresponding monad.

A.3. Generalities on $\lim$

In what follows I will make no distinction between an ordered set S and the usual category it defines; so for a category $\mathbf{C}$ an S-object in $\mathbf{C}$ is a functor $S \to \mathbf{C}$; i.e. a set of objects $C_i, i \in S$, and arrows $C_i \to C_j$ defined for $i \le j$ with the usual compatibilities. Let $\mathbb{Z}$ denote the set of integers with the standard order, and $\Pi = \{(i,j) \mid i \le j\} \subset \mathbb{Z} \times \mathbb{Z}$ with the order induced from $\mathbb{Z} \times \mathbb{Z}$.

Let $\mathbf{A}^{\Pi}$ be the category of Π-objects in $\mathbf{A}$; this is an exact category in the usual way: a short sequence $(X_{i,j}) \to (Y_{i,j}) \to (Z_{i,j})$ is exact iff any corresponding (i,j)'s sequence in $\mathbf{A}$ is exact. Say that an object $X_{i,j}$ of $\mathbf{A}^{\Pi}$ is *admissible* if for any $i \le j \le k$ the corresponding sequence $X_{i,j} \to X_{i,k} \to X_{j,k}$ is short exact. Let $\mathbf{A}_a^{\Pi} \subset \mathbf{A}^{\Pi}$ be the full subcategory of admissible objects. In any short exact sequence if two objects are admissible then the third one is, so $\mathbf{A}_a^{\Pi}$ is an exact category in the obvious way. Clearly $\mathbf{A}$ embeds in $\mathbf{A}_a^{\Pi}$ as the full exact subcategory of objects $X_{i,j}$ with $X_{i,0} = 0$, $X_{1,j} = 0$.

Let ϕ be any order-preserving map $\mathbb{Z} \to \mathbb{Z}$ (so that $\lim_{i \to \pm\infty} \phi(i) = \pm\infty$). Then for any Π-object $X_{i,j}$ we have a Π-object

$$\widetilde{\phi}(X_{i,j}) := X_{\phi(i),\phi(j)};$$

clearly $X \mapsto \widetilde{\phi}(X)$ is an exact functor that preserves $\mathbf{A}_a^{\Pi}$, and $\widetilde{\mathrm{id}}$ is the identity functor. If $\phi \le \psi$, i.e. $\phi(i) \le \psi(i)$ for any i, then we have an obvious morphism of functors $\widetilde{\phi} \to \widetilde{\psi}$.

Define the category $\varprojlim\limits_{\longleftrightarrow} \mathbf{A}$ to be the localization of $\mathbf{A}_a^{\Pi}$ with respect to the morphisms $\widetilde{\phi}(X) \to \widetilde{\psi}(X)$, $X \in \mathbf{A}_a^{\Pi}$, and $\phi \le \psi$ as above. The natural functor $\underset{\longleftrightarrow}{\lim} \colon \mathbf{A}_a^{\Pi} \to \underset{\longleftrightarrow}{\lim} \mathbf{A}$ is surjective on (isomorphism classes of) objects. A morphism $\underset{\longleftrightarrow}{\lim} X_{i,j} \to \underset{\longleftrightarrow}{\lim} Y_{i,j}$ is given by a pair (α, f_α), where $\alpha \colon \mathbb{Z} \to \mathbb{Z}$ is an order-preserving function and $f_\alpha \colon X_{i,j} \to Y_{\alpha(i),\alpha(j)}$ is a compatible system of morphisms; two pairs (α, f_α) and (β, f_β) give the same morphism if f, g give the same maps $X_{i,j} \to Y_{\max\{\alpha(i),\beta(i)\},\max\{\alpha(j),\beta(j)\}}$.

Say that a short sequence in $\underset{\longleftrightarrow}{\lim} \mathbf{A}$ is exact if it is isomorphic to the $\underset{\longleftrightarrow}{\lim}$ of some exact sequence in $\mathbf{A}_a^{\Pi}$. A routine verification of Quillen's axioms shows that this way, $\underset{\longleftrightarrow}{\lim} \mathbf{A}$ becomes an exact category. The functor $\underset{\longleftrightarrow}{\lim}$ is exact; it defines a faithful exact embedding $\mathbf{A} \hookrightarrow \underset{\longleftrightarrow}{\lim} \mathbf{A}$. Any exact functor between $\mathbf{A}$'s induces one between their $\underset{\longleftrightarrow}{\lim}$'s; we also have $\underset{\longleftrightarrow}{\lim}(\mathbf{A}^\circ) = (\underset{\longleftrightarrow}{\lim} \mathbf{A})^\circ$.

Remarks

a. We have $Q \underset{\longleftrightarrow}{\lim} \mathbf{A} = \underset{\longleftrightarrow}{\lim} Q\mathbf{A}$ (where Q is Quillen's Q-construction);

b. If $\mathbf{A} \neq 0$, then $\underset{\longleftrightarrow}{\lim} \mathbf{A}$ is *not* abelian;

c. One can also take $\underset{\longleftrightarrow}{\lim}$'s along any ordered sets with any finite subset having upper and lower bounds.

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